On the Theory of Flexure.

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It is not intended in this discussion to give the *exact* theory of flexure for all materials and shapes of pieces subjected to bending, nor indeed for any one kind of material. The present state of knowledge regarding the internal molecular action developed in any piece of elastic material by the action of external forces, is not such as to enable one to treat any problem of this kind with mathematical rigor if the piece be of finite dimensions. The illustrious Lamé,* however, has remarked that the exact solutions of all problems in natural science are usually obtained by successive approximations, and such has certainly been the case in regard to the theory of flexure.

If the following investigation shall be found to constitute even a short step in the direction of the correct theory, the object of the writer will have been accomplished.

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*An explanation, by the writer, in regard to his aim in this discussion, is very essential in order that the results may not be misunderstood. It is not intended to cover any of the ground gone over so elegantly by St. Venant, Clebsch and others. Their investigations leave nothing to be desired.

It is intended to point out considerations which, it is believed, will account for the great discrepancies existing between the results of the "common theory" and those of experiment. Those considerations apply chiefly to the conditions of stress existing between the elastic limit and rupture, to which the investigations of the authors mentioned above do not apply.

It may easily be shown that the logarithmic law found is not consistent with the equations of condition (4), (5), (6) and (7) for a body of homogeneous elasticity, but those equations do not obtain beyond the elastic limit, nor for bodies that are not homogeneous (and non-homogenity is characteristic of all bodies used by the engineer), nor indeed are they strictly true for homogeneous bodies except for indefinitely small strains. Now indefinitely small strains are by no means those which accompany the application of finite external forces or the existence of finite internal stresses.

Again the researches of M. Tresca, in particular, but also those of Prof. Thurston and others * show that molecules rearrange themselves, to a greater or less extent, when the material in which they exist is subjected to stress for a finite length of time. It is not only possible, but highly probable, that this rearrangement enables the molecules to take such positions as will give the material the greatest possible capacity of resistence.

It is submitted, therefore, that, while it is altogether probable that that condition will exist just before rupture, which, by the principle of least resistance, will subject the material to the least stress, the same law, on the further investigation of strains in either homogeneous or non-homogeneous bodies, *may* be found to hold in the case of such bodies in equilibrium. For that reason some approximate values for the deflection are found which may serve the purpose of (at least) a rough experimental test.

The importance of the bearing of these matters on elastic bodies, is enhanced by the fact that no law of stress whatever can exist in such bodies in equilibrium which may not be supposed to exist in a rigid body.

The arbitrary functions of integration in u, v and w are not all found, for they are not needed for the purposes of the investigation, and a search for them would cause the paper to reach far beyond its proper limits.

[* As, for instance, Eaton Hodgkinson, who, we believe, made accurate determinations in this subject many years before those whose names are above mentioned, having turned his attention to it as early as 1824.—EDS.]

It is assumed, and assumed only in the "Common Theory of Flexure" put forth by Mariotte and Leibnitz, that the intensities of the normal internal stresses parallel to the neutral surface vary directly as the first powers of the normal distances from that neutral surface. This assumption gives results corresponding to experimental ones, with degrees of approximation varying according to the nature of the material and the shape of the piece subjected to bending. Its chief merit, and a very great one, is that it leads to very simple discussions of the cases which ordinarily occur in practice. It ignores, however, the existence of any internal shearing stress, and the formulae deduced for deflection do not involve the distortion which any piece of material suffers when subjected to the action of external forces.

Nevertheless, the method of fixing the position of the neutral surface is correct, since it is based on one of the first principles of statics, *i. e.*, that each of the sums of the components of the internal stresses, taken along three rectangular axes, must be equal to zero. The sum of the component forces of each sign along any axis, and not the sum of the component moments, must be equal to each other when the external forces act in a direction normal to the axis of the beam.

Navier first assumed the equality of the moments, but soon after abandoned the idea and pronounced it erroneous.

The principle just stated, first given by Parent, will be used in the following discussion in the determination of the position of the neutral surface.

Two assumptions will be made, the last only of which, however, as will eventually be shown, tends to give the investigation an approximate character.

The one source of approximation which probably causes the discrepancy between the results which follow and those of experiments is the neglect of lateral contraction and expansion; and those phenomena will be noticed further on.

It will first be assumed that the material has a non-crystalline structure. This is not absolutely necessary, but it emphasizes the proof that the results apply to material of any kind.

The second assumption is this, that the applied bending forces produce no compression at their points of application. This really amounts to supposing the bending to be produced by a single force acting at the proper distance from the section under consideration, while the portion of the beam on the other side of the section is held in position by the requisite forces.

If this assumption were a cause of approximation in the results, those results would not be essentially changed thereby in all ordinary cases of engineering practice as the compression is very slight.

In the case of glass the experiments of the late Mr. Louis Nickerson, C. E., of St. Louis, would seem to show that a high intensity of local pressure at the point where the external force is applied causes the neutral surface to move toward that point through an appreciable distance.

The general equations of equilibrium, however, do not indicate such a result, and there are strong reasons for believing that his experiments may have indicated something different.

The first assumption made renders it possible to make use of Lamé's general equations for homogeneous solids of constant elasticity. These are found on page 65 of his "Leçons sur le *théorie mathématique de l'elasticité des corps solides*," and are the following. Let u, v and w be the actual displacements of any molecule along any assumed three rectangular axes of x, y and z; then N_1, N_2 and N_3 represent the three normal intensities of stresses along these axes respectively, and T_1, T_2 and T_3 the intensities of tangential stresses producing moments around the same axes, *i. e.* T_1 around x, T_2 around y, and T_3 around z. Let λ and μ represent empirical constants depending on the nature of the material, and let $\theta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$. This quantity θ will be recognized as the dilitation per unit of volume. Using this notation, the general equation for a homogeneous solid are

$$\left.\begin{aligned} N_1 &= \lambda\theta + 2\mu\frac{du}{dx}, \quad & T_1 &= \mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right) \\ N_2 &= \lambda\theta + 2\mu\frac{dv}{dy}, \quad & T_2 &= \mu\left(\frac{dw}{dx} + \frac{du}{dz}\right) \\ N_3 &= \lambda\theta + 2\mu\frac{dw}{dz}, \quad & T_3 &= \mu\left(\frac{du}{dy} + \frac{dv}{dx}\right) \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

No demonstration of these equations is given, for it is difficult to conceive of one more elegant or more general than that given by Lamé.

Neglecting the effect of forces emanating from an exterior centre, the conditions of equilibrium are involved in the following equations, also given by Lamé,

$$\left.\begin{aligned} \frac{dN_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz} &= 0 \\ \frac{dT_3}{dx} + \frac{dN_2}{dy} + \frac{dT_1}{dz} &= 0 \\ \frac{dT_2}{dx} + \frac{dT_1}{dy} + \frac{dN_3}{dz} &= 0 \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (2)$$

These are the only equations of condition resulting from the consideration of the principles of statics alone, and are, in general, insufficient to determine the six unknown intensities which enter them.

In the following discussion the piece or beam subjected to bending will be supposed to occupy a horizontal position; the bending forces (including the reäction) will be supposed to act in a direction normal to the axis of the beam; the beam will be supposed straight and uniform in normal section; the axis of x will be taken to be parallel to the axis of the beam; the axis of z will be vertical and the axis of y horizontal and perpendicular to that of x. The axes of z and y will thus be parallel to axes of symmetry of the section, if that section be symmetrical and the beam be properly placed. No other kind of section or position will be considered. In the generality of cases the coefficients of elasticity for tension and compression will be considered equal. In the one or two cases where they are not supposed to be equal, the axis of x will still be taken parallel to the axis of the beam, and not coincident with it.

Now in the case of flexure, generally considered, on account of the distortion of the material subjected to stress, the six stresses N_1, N_2, N_3, T_1, T_2, T_3 actually exist, but in some of the cases taken some of them are equal to zero; in others, some of them are so small that they may be considered differential quantities, *i. e.*, they owe their existence to the indefinitely small difference of the intensities of stresses on two small portions of the material indefinitely close together. The omission of these quantities will evidently produce no essential error in the results, though it is true that it takes from the mathematical exactness of the equations.

Beams whose sections, *i. e.* normal sections, are symmetrical in respect to the axis of y and z will first be considered, and it will be assumed that $N_2 = 0$, $N_3 = 0$, and $T_1 = 0$. It should be stated that the sections considered will not only be symmetrical ones but such that they will not have re-entrant contours.

The case of rectangular sections when N_3 is not equal to zero will be taken up afterwards. It might be treated as existing in all beams if the external forces were so applied that T_1 is still zero, but that is an exceptional case and will not be taken up. T_1 may in reality exist as a very small quantity, in some cases, on account of the variable value of T_2 at the neutral surface.

The equations of condition for equilibrium in these cases, from equations 2, will be the three following:

$$\left.\begin{aligned} \frac{dN_1}{dx} + \frac{dT_3}{dy} + \frac{dT_2}{dz} &= 0, \\ \frac{dT_3}{dx} &= 0, \\ \frac{dT_2}{dx} &= 0, \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (3)$$

or

$$\left.\begin{aligned} (\lambda + \mu)\left(\frac{d^2u}{dx^2} + \frac{d^2v}{dxdy} + \frac{d^2w}{dxdz}\right) + \mu\left(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\right) &= 0, \\ \mu\left(\frac{d^2u}{dxdy} + \frac{dv^2}{dx^2}\right) \qquad &= 0, \\ \mu\left(\frac{d^2w}{dx^2} + \frac{d^2u}{dxdz}\right) \qquad &= 0. \end{aligned}\right\} \quad \ldots \quad (4)$$

Three other equations of condition result from the conditions that N_2, N_3 and T_1 each equal zero. These give in connection with equations (1)

$$\lambda\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + 2\mu\frac{dv}{dy} = 0, \quad \ldots \ldots \quad (5)$$

$$\lambda\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right) + 2\mu\frac{dw}{dz} = 0, \quad \ldots \ldots \quad (6)$$

$$\mu\left(\frac{dv}{dz} + \frac{dw}{dy}\right) = 0. \quad \ldots \ldots \quad (7)$$

These equations, as it will afterwards be seen, aid in the determination of the displacements u, v and w. The last two of equations (3) may be integrated at once, and will give

$$T_3 = f\,(y, z) \quad \ldots \ldots \ldots \quad (8)$$

$$T_2 = F\,(y, z) \quad \ldots \ldots \ldots \quad (9)$$

In which f and F signify any arbitrary functions of y and z whatever; they correspond to the "constants" of integration and must be written because the intensities of the internal stresses are, in general, each functions of x, y and z.

Denoting by $f_y'\,(y, z)$ and $F_z'\,(y, z)$ the partial derivatives of T_3 and T_2, respectively, in respect to the variables indicated, the first of equations (3) may be integrated, and will give

$$N_1 = -x\,[f_y'\,(y, z) + F_z'\,(y, z)] + \Psi\,(y, z) \quad \ldots \quad (10)$$

The quantity $\Psi\,(y, z)$ is any arbitrary function of y and z, and it will now be shown that in general it is independent of y and z, as well as of x, and that in many of the cases of pure flexure it may be put equal to zero.

The direction of action of the stress whose intensity is N_1 is normal to its plane of action, which is a normal section of a fibre parallel to the axis of the beam. Now, since the applied bending forces are perpendicular in direction to the axis of the beam, no part of N_1 can result directly from the forces; that is, they have no component parallel to the fibres subjected to the normal stress N_1.

The stress, whose intensity is N_1, exists *only*, therefore, in consequence of the shearing, or tangential, stresses called into action by the slipping over each other of the fibres parallel to the axis of the beam, or in consequence of T_2 and T_3. The expression for N_1 cannot therefore have a part independent of the quantities T_2 and T_3, except in the case (not of pure flexure) where the beam is subjected to the action of an external force acting in the direction of its own length. The function $\Psi(y, z)$ cannot, therefore, depend on the variables y and z unless they appear raised to the zero power; or, in other words, $\Psi(y, z)$ cannot exist except as a constant, since the integrating equation (10) was made in respect to x. But the case treated is that of pure flexure, in which no external force acts upon the beam in the direction of its own length, and in which, consequently, no part of N_1 can be independent of the tangential stresses T_2 and T_3; hence $\Psi(y, z) = 0$ or c, according as the origin of co-ordinates is at a section of no flexure or not.

Again, differentiate equation (10) in respect to y, there results

$$\frac{dN_1}{dy} = -x\left[f_y''(y, z) + \frac{d(F_z'(y, z))}{dy}\right] + \frac{d\Psi(y, z)}{dy} \quad . \quad . \quad . \quad . \qquad (11)$$

In this equation any value of z may be assumed while y is considered the only variable. Let such a value for z be assumed that the equation will apply to the neutral surface. It will not destroy the force of the reasoning to suppose that surface plane, for if it is not plane the equation of its trace on the plane of normal section of the beam will be $z = f(y)$.

Now, in the neutral surface $N_1 = 0$, $T_3 = 0$ and $F_z'(y, z) = 0$ since T_2 has there its maximum value. Consequently $\frac{dN_1}{dy} = 0$, $f_y''(y, z) = 0$,

and
$$\frac{d\Psi(y, z)}{dy}\, dy = 0 \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

Next, differentiate equation (10) in respect to z, and there results

$$\frac{dN_1}{dz} = -x\left[\frac{d(f_y'(y, z))}{dz} + F_z''(y, z)\right] + \frac{d\Psi(y, z)}{dz} \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

Since z is considered the only variable, such a value for y may be taken that the equation will refer to that portion of a normal section of the beam which lies along the axis of symmetry of the section, for which $f'(y, z) = 0$.

Hence
$$\frac{dN_1}{dz} = -xF_z''(y, z) + \frac{d\Psi(y, z)}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Now $\frac{dN_1}{dz}$ is always a positive quantity, but the function $\Psi(y, z)$ is perfectly arbitrary, and it may be given such a value and sign, if it has real existence as a function of the two variables y and z, that the second member of equation (14) may have a sign contrary to that of its first member, whatever may be the value of $-xF_z''(y, z)$.

In order that equation (14) may be a true one, therefore $\frac{d\Psi(y, z)}{dz}dz = 0$; consequently
$$\frac{d\Psi(y, x)}{dy}dy + \frac{d\Psi(y, z)}{dz}dz = 0; \text{ or, } \Psi(y, z) = c, \quad . \quad . \quad . \quad (15)$$
c being a constant quantity. In the case where the origin is taken at a section of no flexure $c = 0$. Otherwise, at the free ends of a beam, and at sections of contra flexure, there will exist normal stresses parallel to the axis, since a portion of the expression for N_1 would then be independent of x.

There is then established the important equation, when the origin is taken at a section of no bending,
$$N_1 = -x[f_y'(y, z) + F_z'(y, z)]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

It is seen by this equation that N_1 varies directly as x. But in this equation there is apparently involved the condition that one external force only is acting at the distance x from the section under consideration. This arises from the fact that the external forces are assumed to produce no compression at their points of application. It does not affect, however, the generality of the equation, for the last two of equations (3) show that whatever may be the bending moment, the above assumption simply means, it is so produced that the total shearing in any section is equal to that in any other, since $\frac{dT_2}{dx}$ and $\frac{dT_3}{dx}$ both equal zero.

The magnitude of the external force then, is a matter of indifference, only it must be constant for the same beam with any given system of loading.

The normal intensity N_1 is, consequently, proportional to the variable lever arm x of any given constant force which may produce the bending moment to which the

beam is subjected at the section considered; or, in other words, it is simply proportional to the bending moment.

This gives at once a method of expressing N_1 in terms of the bending moment of the external forces, and it will be sometimes convenient hereafter to use it.

Hereafter, also, unless otherwise stated, N, instead of N_1, will be written for the general value of the intensity of the normal stress parallel to the axis.

Let n and M_1 represent the values of N and the external bending moment respectively for any given section, and M the general value of the external bending moment, then, by the principle just stated,

$$N = n \frac{M}{M_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

This is a perfectly general expression whatever may be the position of the origin of co-ordinates.

It will now be necessary to return to the discussion of the general form of equation (16),

$$N = -x\,[f_y'(y, z) + F_z'(y, z)] + c \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

taken in connection with equations (8) and (9).

The functions $f(y, z)$ and $F(y, z)$ are perfectly arbitrary; hence it is sufficient for equilibrium to assign any laws whatever for the variations of the intensities T_2 and T_3, and when T_2 and T_3 are known N at once results from equation (18). There are not, therefore, a sufficient number of equations founded on the principles of statics to insure a solution of the problem. The "Principle of Least Resistance," however, furnishes the wanting condition. Now whatever may be the laws governing the quantities N, T_2 and T_3 there are two conditions which must be fulfilled, *i. e.* the moment of the internal stresses in any section must be equal to the moment of the external forces for the same section, and the total shearing stress in any normal section must equal the sum of the external forces acting on one side of that section. But the second of these conditions is really involved in the first, as will now be shown.

Let $f(z', y') = 0$ be the equation of the perimeter of a normal section of the beam, and $A = \int\int dzdy$ its area. Then, remembering that the coefficient of elasticity for tension is assumed equal to that for compression, the equation

expressing the equality between the moment of the internal stresses of any section, and that of the external bending forces will be

$$2\int_b^{z_1}\int_a^{y'} Nz_2dzdy = -2x\int_b^{z_1}\int_a^{y'} [f'_y(y, z) + F'_z(y, z)]\, z_2dzdy$$
$$+ 2c\int_b^{z_1}\int_a^{y'} z^2dzdy = \frac{1}{2}M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

In this equation z_2 is written for convenience for $(z-b)$, and z_1 represents the maximum value of z. Of course b is the value of z for the neutral surface, and a is the value of y for the vertical axis of symmetry.

The lower limits a and b are taken so that the integration will cover one-fourth of the section, and the resulting moment in the second member will, therefore, be one-half the whole bending moment. Since the axis of z is parallel to the axis of symmetry of section, and since the external forces act parallel to it, the integral $\int\int T_2dydz = \Sigma P$, the sum of the external forces which produce the bending, while $\int\int T_3dydz = 0$; these integrals are supposed to cover the whole section.

Now $\int\int_a^{y_1} f_y'(y, z)\, z_2dzdy = \int (T_3)_a^{y'} z_2dz$; but, considering that part of the section on one side of that axis of symmetry which is parallel to the axis of z, for every positive value of z between the limits of z_1 and b there is also a negative value on the other side of the neutral surface. Hence $\int (T_3)_a^{y'} z_2dz = 0$, and the first term of the second member of equation (18) may be omitted. Again, applying the integrals to the whole surface, $\int\int z_2dzdy$ is simply the statical moment of the surface about an axis passing through its centre of gravity, consequently it is equal to zero, and the last term of the second member of equation (18) may be omitted. Hence

$$2\int_b^{z_1}\int_a^{y_1} Nz_2dzdy = -2x\int_b^{z_1}\int_a^{y'} F_z'(y, z)\, z_2dydz = \frac{M}{2}. \quad . \quad . \quad . \quad (19)$$

$\int F_z'(y, z)\, z_2dz = z_2F(y, z) - \int F(y, z)\, dz$. When $z = z_1$, $F(y, z) = 0$, and when $z = b$, $z_2 = 0$. Consequently $\int_b^{z_1} F_z'(y, z)\, z_2dz = -\int_b^{z_1} F(y, z)\, dz$ and

$$\frac{1}{2}M = 2x\int_b^{z_1}\int_a^{y'} F(y, z)\, dzdy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Equation (20) shows that, *if for any section the bending moment remains the same, the shearing force also will remain constant,* which was to be proved.

If equation (20) be differentiated in respect to x, there results

$$\frac{dM}{dx} = 4\int_b^{z_1}\int_a^{y'} F(y, z)\, dzdy, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

which shows that the first differential coefficient of M in respect to x is equal to the total vertical shearing stress in the section, or the sum of all the external forces acting on one side of the section.

This principle, consequently that involved in equation (20), might have been determined from the fundamental equations of statics.

Now referring to equation (16), on account of the arbitrary character of the functions $f(y, z)$ and $F(y, z)$ the sum of all the *internal stresses developed in any section may have any value whatever* without effecting the equilibrium between the internal and external moments. But the principle of least resistance asserts that *the sum of all the internal stresses developed in any section shall be the least possible consistent with the imposed conditions of equilibrium.*

The only imposed conditions of equilibrium are the constancy of the total shearing or tangential stresses developed in any normal section, and the bending moment of the normal internal stresses about an axis perpendicular to the direction of those tangential stresses. But it has already been shown that the two conditions are equivalent to each other when all the external forces are vertical in direction, the axis of z being vertical also; and when the shearing stresses T_2 and moment about the axis of y are considered.

The equations of condition for the shearing stresses T_3 and moment about the axis of z will be $\int\int T_3 dydz = 0$ and $\int\int Ny_2 dydz = 0$. But these are simply special cases of the general equations $\int\int T_3 dydz = \Sigma P$ and $\int\int Ny_2 dydz = M$, consequently the reasoning applied to equation (18) will bear out the same deductions in this case. The two conditions of equilibrium are therefore involved in the latter equation in both cases.

The problem which now presents itself, therefore, is to find the law governing the intensity N so that there may be the two conditions

$$\int_b^{z_1}\int_a^{y'} Ndydz = \text{minimum}, \quad \ldots \ldots \quad (22)$$

$$4\int_b^{z_1}\int_a^{y'} Nz_2 dydz = M. \quad \ldots \ldots \quad (23)$$

The moment M is, of course, constant for any section while N is a variable function of y and z only, as x, like M, is constant for any section. The equations (22) and (23) may be considered typical since $y_2 = y - a$ may be written for z_2 in equation (23).

If α and t denote two variable parameters, Φ, ϕ and ψ different functions, there may be written generally

$$N = \Phi(\alpha, t), \quad y = \phi(\alpha, t) \text{ and } z = \psi(\alpha, t). \quad \ldots \ldots \quad (24)$$

But in the case under consideration y and z are perfectly independent variables, hence the equations (24) reduce to

$$N = \Phi(\alpha, t),\ y = \phi(\alpha) \text{ and } z = \psi(t). \qquad (25)$$

Consequently the minimum value of the quantity $\int_b^{z_1}\int_a^{y'} N dy dz$ will be found by first considering one variable constant and then the other; or in other words by first considering N a function of y and then of z, or *vice versa.* The equations (22) and (23) then become, when z is considered the only variable,

$$\int_b^{z_1} N dz = \text{minimum}, \qquad (26)$$

$$\int_b^{z_1} N z_2 dz = \text{constant}. \qquad (27)$$

At the neutral surface $N = 0$ and when $z = z_1$ let $N = N_0$; then (26) may take the form

$$\int_b^{z_1} N dz = N_0 z_1 - \int_b^{z_1} N' z dz = \text{minimum, in which } N' = \frac{dN}{dz}.$$

Now finding the minimum value of $\int_b^{z_1} N dz$ is the same as finding the least value of N_0 when $\int_b^{z_1} N dz$ is a constant quantity; the conditional equation (27) holding in both cases. Hence, putting $C = \int_b^{z_1} N dz$, the problem involved in (26) takes the form

$$N_0 = \frac{C}{z_1} + \int_b^{z_1} N' \frac{z}{z_1} dz = \text{minimum}. \qquad (28)$$

Since $\frac{C}{z_1}$ is a constant quantity, the last term of the second member of the above equation is all that need be taken into consideration. If a' is a constant, then let S denote the integral of which the absolute minimum is to be found. This function S is obtained by the principles of the Calculus of Variations, by multiplying the conditional equation (27) by a' and adding the result to the variable part of equation (28). These operations give

$$S = \int_b^{z_1} \left[N' \frac{z}{z_1} + a' N (z - b) \right] dz. \qquad (29)$$

The methods of the Calculus of Variations must be applied to this definite integral in order to determine the character of the function N which will give it its least value. The following system of notation is that used in the work of J. A. Serret on the Calculus:

$$V = N' \frac{z}{z_1} + a' N(-zb), \quad Z = \frac{dV}{dz} = \frac{N'}{z_1} + a' N,$$

$$Y = \frac{dV}{dN} = a'(-zb), \qquad Y' = \frac{dV}{dN'} = \frac{z}{z_1}.$$

The condition for a minimum is the following:

$$Y - \frac{dY'}{dz} = 0, \text{ or } Y = \frac{dY'}{dz}.$$

Now, since $dz = d\,(z - b)$, there results for the complete differential of V

$$dV = Zd(z - b) + YdN + Y'dN'. \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

But if this equation be integrated, it is evident the determinate part of the integral of the first term of the second member will be equal to that of the first member, hence

$\int (YdN + Y'dN') = c$; or, from the conditions for a minimum,

$$\int (N'dY' + Y'dN') = c;$$

$$\therefore N'Y' = c = \text{constant}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

Since $$Y' = \frac{z}{z_1} \text{ and } N = \frac{dN'}{dz},$$

$$dN = z_1 c\,\frac{dz}{z} \quad \therefore N = z_1 c \log z + c'. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

Hence the curve representing the law of variation of N is a logarithmic one, and, since the value of z for the neutral surface is b, if b is taken equal to unity, c' will be zero for this case. The value for N, therefore, for a vertical plane passing through the axis of the beam will be

$$N_1 = z_1 c \log z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

The symbol "*log*" refers, of course, to Naperian logarithms. Now, for any part of the beam z_1 must be replaced by z', since $f\,(z', y') = 0$ is the equation of the perimeter of the section. Equation (33), therefore, for any strip of elements parallel to the axis of z and at any distance y from the origin, will take the form

$$N = z'c \log z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Let N_0' represent the value of N for any point of the perimeter of the section, except that one for which $z = e_1$; and for that point let N_0 be written, it will represent the greatest intensity of stress in the section. When $z = z'$, $N = N_0'$; consequently, for equation (34), $c = \dfrac{N_0'}{z' \log z'}$,

$$\therefore N = \frac{N_0'}{\log z'} \log z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

When $z = z_1$, $N_1 = N_0$, $\therefore c = \dfrac{N_0}{z_1 \log z_1}$ for equation (33), and

$$N_1 = \frac{N_0}{\log z_1} \log z. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

From what has already been said, in regard to the general equations of condition for moments and shearing stresses in horizontal planes, between equations (21) and (22), it is evident that the same steps precisely would have to be taken in order to determine the law of variation of N with y, as were taken to determine the connection between N and z. In fact, since y and z are considered variable only in turn, y may be written for z in the general operations for determining the least value of the definite integral S. Hence the typical equation for N may be written in terms of y,

$$N = \frac{N_1}{\log y_1} \log y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

The quantity y_1 denotes the half width of the beam added to a. But, as was done in the case of z, a is assumed to be equal to unity in determining the constant $\dfrac{N'}{\log y_1}$.

Now, in writing the equation (37) there is virtually assumed to be a surface of no stress of the kind N at the distance $(y_1 - 1)$ from the vertical axis of symmetry of the section. In other words, referring to Fig. 1, O' is really the assumed origin and HK the supposed position of the axis of z, while the surface of no stress touches the beam at m. The greatest value of N, therefore, in any horizontal plane, is N_1 found in the vertical axis of symmetry of the beam. The point O is at the distance unity on one side of, and below, the centre C of the section; and it is most convenient to take that point for the origin of co-ordinates. $O'O$ is equal to $(y_1 + 1)$ and $O'F$ is the y of equation (37). This latter quantity in terms of OF, the new y, will be $(y_1 + 1 - OF)$ $= (y_1 - y + 1)$. Consequently, equation (37) takes the form

$$N = \frac{N_1}{\log y_1} \log\,(y_1 - y + 1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

When $y = BD = y'$ then $N = N_0'$,

$$N_0' = \frac{N_1}{\log y_1} \log\,(y_1 - y + 1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

N_1 is determined by equation (36), but z' must be written for z in that equation, then

$$N_1 = \frac{N_0}{\log z_1} \log z', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

$$\therefore\ N_0 = \frac{\log z'}{\log z_1}\,\frac{N_0}{\log y_1} \log\,(y_1 - y' + 1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Substituting in equation (35), there results

$$N = \frac{N_0}{\log z_1} \frac{\log z}{\log y_1} \log (y_1 - y' + 1) . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

This gives the general value of N in terms of the greatest intensity N_0 of the entire section. If $z = z'$, then, by referring to equations (38) and (40) it is seen that $N = N'_0$. If $z = 1$ the equation (42) refers to the neutral surface and $N = 0$.

If $y' = 1$, then the vertical axis of symmetry is referred to, and

$$N = N_1 = \frac{N_0}{\log z_1} \log z.$$

Before passing on farther in the analytical discussion of the problem, it will be well to consider the form of the double curved surface which represents graphically the law of variation of the intensity N.

The closed curve in Fig. 1 represents a normal section of the beam, O being the origin of co-ordinates. Now if normal lines be drawn at each point of the section of Fig. 1 whose lengths represent intensities, N, at the different points, a double curved surface will enclose their extremities from which logarithmic curves, represented by the equations already given, will be cut by vertical and horizontal planes. The shaded portion of Fig. 2 represents a section cut by a vertical plane passed through the axis of the beam, and equation (36) is the equation to its perimeter. The shaded portion of Fig. 3 is a horizontal section made by a plane passed through RS, Fig. 1; the general equation for which is equation (38). CD of Fig. 3 is equal to FH of Fig. 2. All vertical planes will cut sections similar to that in Fig. 2; these sections will have for their equation, equation (35). All horizontal planes will cut sections similar to that in Fig. 3 and equation (38) will be the general equation to their perimeters.

The tangent of the angle made by the curve at C, Fig. 2, with BC is equal to $\frac{dN_1}{dz} = \frac{N_0}{\log z_1}$, since for that point $z = 1$. The general value for the tangent is $\frac{dN_1}{dz} = \frac{N_0}{\log z_1} \frac{1}{z}$. Hence the curve becomes parallel to BC at an infinite distance from the origin, and has a horizontal asymptote passing through the origin. The same reasoning applies to the curves of the other sections.

$\frac{d^2N}{dz^2} = - \frac{N_0}{\log z_1} \frac{1}{z^2}$; hence the vertical curves are concave towards the axis of

z. For the same reason the horizontal curves are concave towards the axis of y. The whole surface therefore is concave towards the plane of section.

Nothing has been said in regard to the determination of the position of the neutral surface, except the statement made in the beginning, which would make it a plane before flexure passing through the centre of gravity of a normal section, on the supposition that the coefficients for tension and compression are equal to each other. The true principle has so long been recognized that it is not necessary to speak farther of it here.

Referring to equations (17) and (42), it is evident that the general value for the intensity N will be

$$N = \frac{M}{M_1} \frac{N_0}{\log z_1} \frac{\log z}{\log y_1} \log (y_1 - y' + 1) \; . \; . \; . \; . \; . \; . \qquad (43)$$

It is also evident that equation (43) may be so written as to apply to a horizontal plane at the distance z' from the origin; it will then take the form

$$N = \frac{M}{M_1} \frac{N_0}{\log z_1} \frac{\log z'}{\log y_1} \log (y_1 - y + 1) \; . \; . \; . \; . \; . \; . \qquad (44)$$

Although this is deduced immediately from equation (43), it may be demonstrated in precisely the same manner as was that equation.

The moment of resistance of the beam may now be easily written, though the integration involved may yet be found impossible in some cases and intricate in all but rectangular beams.

It is well known that the tangential stresses existing on the sides of a small parallelopipedical portion of any material constitute a system of forces in equilibrium. Consequently, the moment of resistance in any section will be the sum of the small moments $Ndydz \, . \, (z - 1)$. The lever arm of each of the small forces $Ndydz$ is $(z - 1)$, because the centre of moments is taken in the neutral surface and the origin of co-ordinates is at the distance unity below that surface.

Since the normal sections of all the beams considered are symmetrical and without re-entrant outlines, the following equation at once results:

$$\frac{1}{4} M = \frac{N_0}{\log z_1 \, \log y_1} \int_1^{y_1} \int_1^{z'} \log (y_1 - y' + 1) \log z \, . \, (z - 1) \, dzdy'. \qquad (45)$$

N_0 is, of course, the greatest intensity in the given section. Since y is an independent variable, dy may be taken equal to dy'.

Now, $\int_1^{z'} (z - 1) \log z \, . \, dz = \frac{1}{2} z'^2 \log z' - \frac{1}{4} z'^2 - z' \log z' + z' - \frac{3}{4}$, and

$z' = f(y')$ is the equation to the perimeter of the section. Consequently

$$\frac{1}{4} M = \frac{N_0}{\log z_1 \log y_1} \int_1^{y_1} \left(\frac{1}{2} \overline{f(y')}^2 \log f(y') - \frac{1}{4} \overline{f(y')}^2 - f(y') \log f(y') + f(y') - \frac{3}{4} \right) \log (y_1 - y' + 1)\, dy'. \quad . \quad . \quad . \quad . \quad . \quad (46)$$

This intricate expression reduces to a much simpler one for beams of rectangular section. Equation (45) might have been written in terms of z' and y, in which case, equation (46) would have been found in terms of z', but would not be in as convenient shape.

The general values of the displacements u, v and w may now be approximately determined. It must be remembered that these displacements will only exist when N_2, N_3 and T_1 are each equal to zero. From the equation (5) and (6) there results the relation

$$\frac{dv}{dy} = \frac{dw}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

Then, from either equation (5) or equation (6),

$$\frac{du}{dx} = -\frac{2(\lambda + \mu)}{\lambda} \frac{dw}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

But, from equation (1),

$$N = \lambda \left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} \right) + 2\mu \frac{du}{dx} = 2\lambda \frac{dw}{dz} + (\lambda + 2\mu) \frac{du}{dx}. \quad . \quad . \quad (49)$$

Substituting from (48) in (49), there results

$$N = -\frac{2\mu}{\lambda} (3\lambda + 2\mu) \frac{dw}{dz}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

From equation (50), in connection with equation (47), there at once result

$$w = -\frac{\lambda}{2\mu(3\lambda + 2\mu)} \int N dz, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

$$v = -\frac{\lambda}{2\mu(3\lambda + 2\mu)} \int N dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

Equation (50) gives also the relation

$$-\frac{2(\lambda + \mu)}{\lambda} \frac{dw}{dz} = \frac{(\lambda + \mu)}{\mu(3\lambda + 2\mu)} N = EN.$$

Combining this with equation (48), there results

$$u = E \int N dx. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

The coefficient of elasticity $E = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}$ is written as M. Lamé uses it, *i. e.* so as to represent the strain for each unit of stress. For wrought iron

E would have an average value of, say, $\frac{1}{26000000}$. The relation between E, μ and λ will be found given in the work of M. Lamé before mentioned. In finding the value for w, N is to be taken in terms of y' and z, and in determining v, it is to be taken in terms of z' and y. Substituting the values for N, there result the following expressions for u, v and w:

$$w = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\frac{M}{M_1}\frac{N_0}{\log z_1 \log y_1}\log(y_1 - y' + 1)(z\log z - z) + f(x, y),$$

$$v = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\frac{M}{M_1}\frac{N_0}{\log z_1 \log y_1}\log z'\left[(y_1 - y + 1) - (y_1 - y + 1)\log(y_1 - y + 1)\right] + f(x, z),$$

$$u = \frac{EN_0}{M_1 \log z_1 \log y_1}\log(y_1 - y' + 1)\log z \int M dx + f(z, y).$$

The functions $f(\ )$ must be added in each integration because w, v and u are each functions of the three independent variables x, y and z.

Let Δ be the deflection of the upper surface of the beam at *any* point; then when $z = z'$, $w = \Delta$. In the vertical plane of symmetry for the beam $v = 0$; hence when $y = 1$, v will equal zero.

The term $f(z, y)$ in the expression for u will depend upon the configuration of that section of the beam in which the origin of co-ordinates is located, it expresses the displacement in the direction of x for that section. If that section remains plane and vertical after flexure $f(z, y)$ will reduce to zero, or a constant, and it will always be equal to zero for the neutral surface if it be assumed that the section containing the origin suffers no movement, as a whole during flexure. For any other point not in the neutral surface its value will depend on the distribution of tangential stress in the section where the origin is found, and its value is not easy to determine. In all cases of ordinary experience it is a very small quantity compared with the other parts of the deflection, and essentially no error will be committed by its omission; such an omission will be made in equation (56).

By introducing the given conditions the values of w, v and u will be written as follows:

$$w = \frac{\lambda}{2\mu(3\lambda+2\mu)}\frac{M}{M_1}\frac{N_0}{\log z_1 \log y_1}\log(y_1 - y' + 1)(z'\log z' - z' - z\log z + z) + \Delta, \quad (54)$$

$$v = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\frac{M}{M_1}\frac{N_0}{\log z_1 \log y_1}\log z'\left[y_1\log y_1 + (1 - y) - (y_1 - y + 1)\log(y_1 - y + 1)\right], \quad (55)$$

$$u = \frac{EN_0}{M_1 \log z_1 \log y_1}\log(y_1 - y' + 1)\log z \int \Sigma P(x_1 - x)\, dx. \quad (56)$$

Now, equation (54) has been written involving Δ, the deflection of the upper surface of the beam, but it must be remembered that w in the values of N, T_2 and T_3 represents simply the displacements in the given section, or that which is caused by the stresses acting and not by any bodily movement of any portion of the beam. In writing the value of T_2, therefore, in equation (57), Δ must be omitted in equation (54). Otherwise, it would be true, as a general principle, that the shearing stress in any section is dependent on the deflection Δ, which is evidently not true. In equation (56), x_1 is the co-ordinate of the section under consideration, and x is the general value of the abscissa of the point of application of the force P; or, in other words, $M = \Sigma P\,(x_1 - x)$.

It is seen from the value of w immediately preceding equation (54), that the deflection of the neutral surface at any point is independent of the variable z, and is a function of the independent variables x and y. This result shows that the neutral surface is not a cylindrical one after flexure, although it is symmetrical in reference to a vertical plane of symmetry for the beam. The neutral surface, then, is a surface of double curvature for all beams except those with rectangular sections, for which it is cylindrical.

Since Δ depends on x and y, and not on z, the deflection of the neutral surface may be determined if the maximum intensity of direct stress N_0 is known for the given section, as will be seen hereafter.

In equations (1) there are given general values for the intensities of the tangential stresses T_2 and T_3 in terms of u, v and w. Using equations (54–56), the two following equations are deduced, remembering what has already been said in regard to equation (54):

$$T_2 = \frac{M_1 \log z_1 \log y_1}{\log\,(y_1 - y' + 1)}\,\frac{2\,(3\lambda + 2\mu)}{\lambda N_0}\,(z' \log z' - z' - z \log z + z)\,\Sigma P$$
$$+ \frac{(\lambda + \mu)}{(3\lambda + 2\mu)}\,\frac{N_0}{M_1 \log z_1 \log y_1}\,\frac{\log\,(y_1 - y' + 1)}{z}\int \Sigma P\,(x_1 - x)\,dx\,, \quad . \quad . \quad . \quad (57)$$

$$T_3 = -\frac{\lambda N_0}{2(3\lambda + 2\mu)}\,\frac{\log z'}{M_1 \log z_1 \log y_1}\,[y_1 \log y_1 + (1 - y) - (y_1 - y + 1) \log\,(y_1 - y + 1)]\,\Sigma P$$
$$- \frac{\lambda + \mu}{(3\lambda + 2\mu)}\,\frac{N_0 \log z'}{M_1 \log z_1 \log y_1}\,\frac{\int \Sigma P\,(x_1 - x)\,dx}{(y_1 - y + 1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

It has been assumed that $f\,(z, y)$ in the value of u is equal to zero for both equations (57) and (58). If this cannot be admitted, then $\mu\,\dfrac{df\,(z, y)}{dz}$ is to

be added to the second member of equation (57), and $\mu \frac{df(z, y)}{dy}$ to that of equation (58).

If the partial differential coefficients of T_2 and T_3 be taken in respect to z and y, the two following equations will result after having substituted from the general values of N:

$$\frac{dT_2}{dz} = -\frac{(\lambda + \mu)}{(3\lambda + 2\mu)} \cdot \frac{\int M dx}{M} \cdot \frac{d^2 N}{dz^2} - \frac{\lambda}{2(3\lambda + 2\mu)} \frac{\Sigma P}{M} N, \quad . \quad . \quad . \quad (59)$$

$$\frac{dT_3}{dy} = \frac{(\lambda + \mu)}{(3\lambda + 2\mu)} \cdot \frac{\int M dx}{M} \cdot \frac{d^2 N}{dy^2} - \frac{\lambda}{2(3\lambda + 2\mu)} \frac{\Sigma P}{M} N. \quad . \quad . \quad . \quad . \quad (60)$$

Now, from equation (16), it is seen that:

$$\frac{dN_1}{dz} = \frac{\Sigma P}{M} N = -\left(\frac{dT_2}{dz} + \frac{dT_3}{dy}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

But equation (60) shows that equation (61) is only true when $\lambda = -\mu$ or $\mu = -\lambda$; or when $E = 0$, since $E = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)}$; or when the material is rigid so far as tensile and compressive stresses are concerned. Lateral displacements due to shearing stresses, however, may be supposed to exist.

Equations (1) give the general values of the intensities N_1, T_2 and T_3, but in order that equilibrium may exist they must be subject to the conditions of equations (4), which are perfectly independent of the equations (1). In fact equations (4) are founded on the first principles of statics and are perfectly independent of the nature of the material in which stress may exist. This matter will be specially noticed farther on.

The equations (59), (60) and (61) show that the distribution of the shearing or tangential stresses in the beam subjected to flexure is independent of the quantities λ and μ, and is the same whether the beam be supposed rigid or elastic with a finite value of E. Making $\lambda = -\mu$ therefore in equations (57) and (58), there results

$$T_2 = \frac{N_0}{2M} \frac{\log(y_1 - y' + 1)}{\log z_1 \log y_1} (z' \log z' - z' - z \log z + z) \Sigma P, \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

$$T_3 = -\frac{N_0}{2M_1} \frac{\log z'}{\log z_1 \log y_1} [y_1 \log y_1 + (1 - y) - (y_1 - y + 1) \log(y_1 - y + 1)] \Sigma P. \quad . \quad (63)$$

These are the true values of the intensities of the tangential stresses, and it will hereafter be shown that $4 \int_1^{y_1} \int_1^{z'} T_2 dz dy = \Sigma P$, as should be the case.

It has already been shown that $\frac{dM}{dx} = \Sigma P$, consequently equations (62) and (63) may be written in terms of $\frac{dM}{dx}$, and it will sometimes be convenient to use them in that form hereafter.

There is an apparent anomaly in the fact that equations (57) and (58) are the expressions derived directly from the general values of T_2 and T_3 in equations (1), while equations (62) and (63) are the true values of these intensities. The explanation is found in what has already been said in regard to the intensities being the same as in a material for which E is equal to zero. Equations (62) and (63) also show that T_2 and T_3 are independent of x, except in so far as that variable may enter the summation ΣP, which is consistent with one of the first general equations of condition.

From equations (62) and (63) the following results flow: if $z' = 1$, $T_3 = 0$ for all values of y; if $z = z' = 1$, $T_2 = 0$, and if $z = z'$ only, $T_2 = 0$; if $y = 1$, $T_3 = 0$. These results are as they should be, and might have been anticipated.

Another method of deducing the displacement w, in which Δ will represent the deflection of any point of the neutral surface, is the one which follows. It is somewhat more convenient in the treatment of beams with rectangular cross sections. Let Δ then represent the deflection of any point of the neutral surface. When $z = 1$, in the value of w immediately preceding equation (54), $w = \Delta$, hence

$$f(x, y) = \Delta - \frac{\lambda}{2\mu(3\lambda + 2\mu)} \frac{M}{M_1} \frac{N_0}{\log z_1 \log y_1} \log(y_1 - y' + 1). \quad . \quad . \quad (64)$$

If Δ', therefore, represents the general value of the deflection, there will result, instead of equation (54),

$$\Delta' = - \frac{\lambda}{2\mu(3\lambda + 2\mu)} \frac{M}{M_1} \frac{N_0}{\log z_1 \log y_1} \log(y_1 - y' + 1) \,.\, (z \log z - z + 1) + \Delta. \quad (65)$$

Now, let Δ'' represent the value of Δ' when $z = z'$, then the quantity w, which is to be used in writing the value of T_2, will be equal to $\Delta' - \Delta''$. Hence

$$w = \frac{\lambda}{2\mu(3\lambda + 2\mu)} \frac{M}{M_1} \frac{N_0}{\log z_1 \log y_1} \log(y_1 - y' + 1) \,.\, (z' \log z' - z' - z \log z + z). \quad (66)$$

Equation (66) is the same as equation (54) with Δ omitted from the latter. The consequences indicated by equations (56) and (57) might be deduced more simply, perhaps, from equations (65) and (66) than from equation (54) and the one preceding it.

It is to be noticed here that the expressions for the intensities N and T_2, as well as T_3, are perfectly general, although the original equations of condition were based on the supposition that the bending moment should be produced by a single force or a couple. Their generality is due to two facts: a given amount of shearing stress will always be distributed over the same section in precisely the same way, whether that amount is made up of reaction at a point of support combined with external loads imposed between that point and the given section, or whether that amount is equal to a single force P hung at the free end of a beam; and a bending moment M may be produced by a single force P or by a number of forces whose combined effect produces the given moment, and the distribution of the direct stresses of tension and compression will be precisely the same in each case.

By reference to equation (46) it is seen that the quantity $\frac{N_0}{M}$, or $\frac{N_0}{M_1}$ (the intensity and the moment must belong to the same section), is not altogether dependent on the form of the cross section, since the quantities $\log z_1$, $\log y_1$ and $\frac{3}{4}$ enter the expression for M, *but is a constant quantity for the same beam.* In like manner $\frac{N}{M}$ is constant for the same beam, if N is always taken at a point whose co-ordinates y and z are the same in the different sections.

It is evident that the maximum value of T_2 will be found at the centre of any section; consequently its value will be determined by making $z' = z_1$, $y' = 1$ and $z = 1$ in equation (62). Denoting the maximum value of T_2 by T_m, there results

$$T_m = \frac{N_0}{2M_1} \frac{1}{\log z_1} (z_1 \log z_1 - z_1 + 1) \Sigma P. \quad . \quad . \quad . \quad . \quad . \quad (67)$$

Let A be the area of the section of the beam to which equation (67) applies, then the mean shearing intensity in any section will be $\frac{\Sigma P}{A}$. The ratio, therefore, between the maximum and mean intensities of shear in any section will be

$$T_m \frac{A}{\Sigma P} = \frac{N_0}{2M_1} \frac{(z_1 \log z_1 - z_1 + 1)}{\log z_1} A. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

This expression is not constant for the same form of cross section, but is constant for the same beam.

When ΣP is equal to zero, both T_2 and T_3 reduce to nothing. This case exists where a portion of a beam is bent by a couple and where evidently the

curve of flexure must be circular, since N cannot vary if T_2 and T_3 are both equal to zero, as equation (10) shows. This is one of the special cases in which $\Psi\ (y, z)$ is a constant.

The expressions for T_2 and T_3 show how the shearing stress is supposed to be distributed at the free ends of beams and at sections of contraflexure, and furnishes the data for determining the quantity $f\,(z, y)$ in the value for the longitudinal displacement u. As, however, it is of little practical value it will not be determined. The reaction, therefore, at the free ends of beams and external forces acting at sections of contraflexure are supposed to be so distributed over the sections of the beam that $\int\int T_2 dydz = \Sigma P$.

The deflection of the beam is next to be determined, and it has already been shown that that part of it, Δ, due to the bodily movement of a portion of the beam is not a function of z, but is dependent only on x and y. It varies of course with the half depth of the beam, or with what amounts to the same thing, the quantity z_1.

The movement of the molecules of the material, relatively to each other, in any given section, is to be determined by the value of w from equation (66) which was used in fixing the value of T_2.

Let Δ_1 represent the deflection of any point of the *upper surface* of the beam. From what has already been said in regard to Δ, and, from the general conditions of the problem, it is clear that this depends only on the lengthening or shortening of the exterior fibres in the upper surface of the beam.

The upper surface of the beam is here mentioned, although "the lower surface" might have been written just as well.

The rate $\frac{du}{dx}$ of the longitudinal displacement at any point, is due to the intensity N, or N_0', if that point is in the exterior surface. Let u_0 be the value of u for any point where N_0' exists, then $\frac{du_0}{dx} = EN_0'$. The coefficient of elasticity E, of course, represents the rate of lengthening or shortening of a fibre at any point for each unit of N_0'. Now, if the beam be divided into indefinitely thin rectangular portions by vertical planes parallel to the axis of the beam, each portion may be supposed to be an actual rectangular beam subjected to such a moment that the greatest intensity of direct stress is equal to N_0' at the given section. The sum of all these elementary moments for

any section will be equal to the moment to which the original beam is subjected; and the sum ΣP of all the external forces acting on all the elementary beams for the same sections, will be equal to the sum ΣP for the original beam at the same section. From this it follows that the deflections of different points in the exterior surface have different values; also, that the deflection of any such point is precisely the same as that which a rectangular beam would have if the circumstances of loading and length were the same in each case, and if the depth of the given beam at the given point were equal to the depth of the supposed rectangular beam; which conditions make N_0' the same for each.

These considerations show that the deflection of that point in the exterior surface of a beam which is farthest from the neutral surface, is independent of the form of cross section, and is the same as that of a rectangular beam in the same circumstances; which results also from the "common theory."

In Figure 5, let AB be a portion of the line of intersection of a longitudinal plane with the neutral surface, and C, the centre of curvature of AB. BD is parallel to AC, then $FD = AB = 1$. Let $AC = r$, then will $DE = \frac{du_0}{dx}$. From similarity of triangles, since $AF = BD = (z' - 1)$, in general

$$\frac{\frac{du_0}{dx}}{z'-1} = \frac{1}{r} = \frac{EN_0'}{z'-1} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

This gives the value of the reciprocal of the radius of curvature in a longitudinal plane at any point, and its general form and method of demonstration is precisely that used in the common theory. If there be written that approximate value of $\frac{1}{r} = \frac{d^2y}{dx^2}$, which was introduced by Navier, and $N_0' = f(M)$, there will result

$$\frac{d^2y}{dx^2} = \frac{Ef(M)}{(z'-1)} \cdot \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

The y in equation (70) is not the one heretofore used, but represents the deflection due to the displacement EN_0', and taken between the proper limits, is equal to Δ_1.

Before developing this matter of the deflection farther, it will be well, for the reasons already given, to find equations for beams of rectangular sections.

In order to make the general value of N in equations (43) or (44) apply to rectangular beams, it is only necessary to put y or y' equal to unity and write z for z'. Performing these simple operations, there will result

$$N = \frac{M}{M_1}\frac{N_0}{\log z_1}\log z\,. \quad (71)$$

The same substitutions made in equations (62) and (63) give

$$T_2 = \frac{N_0}{2M_1}\frac{(z_1 \log z_1 - z\log z - z_1 + z)}{\log z_1}\Sigma P, \quad (72)$$

$$T_3 = 0, \quad (73)$$

The result shown in equation (73) was to have been anticipated.

Equations (71) and (72) might have been established directly by a course of reasoning precisely similar to that followed for a beam of any symmetrical but solid section, in which case, in addition to equations (5), (6) and (7), there would have been the one indicating that $T_3 = 0$.

Let b be the breadth of the rectangular beam, then equation (45) will reduce to the form

$$\begin{aligned} M &= 2b\,\frac{N_0}{\log z_1}\int_1^{z_1}(z-1)\log z\,.\,dz \\ &= 2b\,\frac{N_0}{\log z_1}\left[\frac{1}{2}z_2\log z - \frac{1}{4}z_2 - z\log z + z\right]_1^{z_1} \\ &= 2b\,\frac{N_0}{\log z_1}\left(\frac{1}{4}z_1^2\log\frac{z_1^2}{e} - z_1\log\frac{z_1}{e} - \frac{3}{4}\right). \quad (74)\end{aligned}$$

In equation (74) e is the base of the Naperian system of logarithms, and N_0 is the greatest intensity of direct stress in the given section. The quantity $\frac{M}{b}$ is the bending moment for each unit of breadth, and it is seen from equation (74) that $\frac{bN_0}{M}$ is a constant quantity for all rectangular beams of the same depth.

The sum of all the shearing stresses in the section ought to be equal to ΣP. Hence, from the general expression following equation (63),

$$4b\int_1^1 T_2 dz = \frac{2bN_0}{M_1\log z_1}\left(\frac{1}{4}z_1^2\log\frac{z_1^2}{e} - z_1\log\frac{z_1}{e} - \frac{3}{4}\right)\Sigma P.$$

But by equation (74) the second member of this equation is equal to $-\Sigma P$; hence

$$4b\int_1^{z_1} T_2 dz = \Sigma P. \quad (75)$$

It has already been stated that the assumption $N_3 = 0$ is no cause of error in the results for beams of rectangular section; it will next be shown

that such is the case. For rectangular beams the equations (2) reduce to the following:

$$\frac{dN_1}{dx}+\frac{dT_2}{dz}=0,$$

$$\frac{dT_2}{dx}+\frac{dN_3}{dz}=0.$$

The three intensities N_1, T_2 and N_3 are each functions of z and x only. The "principle of least resistance" determines N_1 at once as given by equation (71); T_2 at once follows in equation (72). The second of the above equations in connection with equation (72) gives

$$\frac{dN_3}{dz}=-\frac{dT_2}{dx}=-\frac{N_0}{2M_1}\left\{\frac{(z_1\log z_1-z_1)-(z\log z-z)}{\log z_1}\right\}\frac{d\Sigma P}{dx}.$$

The quantity $\frac{d\Sigma P}{dx}$ is the intensity of external vertical pressure at any point; denote it by $-p$. Then

$$N_3=+\frac{N_0 p}{2M_1\log z_1}\left\{(z_1\log z_1-z_1)\,z-\frac{1}{2}z^2\log z+\frac{3}{4}z^2\right\}+f(x).$$

When $z=z_1$, $N_3=-p$, hence

$$N_3=-\frac{N_0 p}{2M_1\log z_1}\left\{\frac{1}{2}z_1^2\log z_1-\frac{1}{4}z_1^2-(z_1\log z_1-z_1)\,z+\frac{1}{2}z^2\log z-\frac{3}{4}z^2\right\}-p.$$

These values of the intensities N_1, T_2 and N_3 satisfy the two simultaneous equations of condition given above.

The assertion which immediately follows equation (63) may now be proved without difficulty. Equation (62) may be put under the following form:

$$T_2=\frac{\Sigma P}{2M_1}\left\{\frac{N_0}{\log z'}\;\frac{\log z'\log(y_1-y'+1)}{\log z_1\log y_1}\,(z'\log z'-z'-z\log z+z)\right\};$$

or, by equation (41),

$$T_2=\frac{\Sigma P}{2M_1}\left\{\frac{N_0'}{\log z'}\,(z'\log z'-z'-z\log z+z)\right\}\;.\;.\;.\;.\;.\quad(76)$$

Now, the given beam is equivalent to an indefinite number of elementary beams of the constant or variable width $2dy$ (corresponding to b), and having the variable depth $2\,(z'-1)$. Hence, the integral $4\int\int T_2dzdy$ may be put in the following form:

$$4\int_1^{y_1}\int_1^{z'}T_2dzdy=\frac{\Sigma P}{M_1}\,\Sigma\left\{\int_1^{z'}\frac{N_0'}{\log z'}\,(z'\log z'-z'-z\log z+z)\right\}2dy.\;\;.\;\;.\quad(77)$$

But from the equations immediately preceding equation (75) it is evident that that part of the second member of equation (77) which follows the second

Σ is the general expression for the moment to which any elementary rectangular beam is subjected; hence the sum of all those moments denoted by Σ must be equal to (M_1). Hence

$$4\int_1^{y_1}\int_1^{z'} T_2 dydz = \Sigma P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (78)$$

It is evident from the preceding that the expression, as a general one, $\frac{\Sigma P}{M_1}$ is the same for all the elementary beams and the original beam itself. M_1 and ΣP belong, of course, to the same section.

The subject of deflection can now be resumed. Let Y_1 represent the definite integral in equation (46), then from equation (41) there results

$$N_0' = \frac{\log z' \log (y_1 - y' + 1)}{4Y_1} M = f(M). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (79)$$

Equation (79) gives the value of $f(M)$ in equation (70) for the general case. As has already been shown, however, it may be only necessary to find the function for a rectangular section in which $b = 1$ and $z_1 = z'$. To determine, therefore, that part of the deflection which is denoted by Δ_1, find the value of N_0' from equation (79) and substitute it in equation (70), then, if Z_1 be put for $\frac{\log z' \log (y_1 - y' + 1)}{4Y_1(z'-1)}$, that equation will give

$$y = \Delta_1 = EZ_1 \int\int Mdx^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (80)$$

This is precisely the expression given by the "common theory" if $\frac{1}{I}$ (I being the moment of inertia of the cross section) be written for Z_1. The ordinary values for y may therefore be used in equation (80) by inserting in the formulæ of the "common theory" Z_1 for $\frac{1}{I}$.

If N_0' is known for any point, then by equation (74)

$$N_0' = \frac{\frac{1}{2}\log z'}{\left(\frac{1}{4}z'^2 \log \frac{z'^2}{e} - z' \log \frac{z'}{e} - \frac{3}{4}\right)} M = ZM. \quad . \quad . \quad . \qquad (81)$$

That which is represented by Z is evident from the equation. There will then result as before

$$y = \Delta_1 = EZ \int\int Mdx^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (82)$$

The remarks following equation (80) apply also, as is evident, to equation (82).

Equations (80) and (82) give that part of the deflection which is due to the bodily moment of a portion of the beam and which is caused by the longitudinal displacement u. Another part is that due to the shearing stress T_2 at the neutral surface, which causes layers, made by vertical planes normal to the axis of the beam, to slip by each other to a greater or less extent.

It should be understood that when the "deflection of the beam" or "total deflection" is spoken of, the neutral surface is what is referred to.

That portion of the total deflection which is due to T_2, or the displacement (vertical) in any given section is given by equation (66) after making $\lambda = -\mu$. Let M_0 be the value of M at the point from which the deflection is measured, and w_1 represent this part of the total deflection, then

$$w_1 = \frac{M - M_0}{2 M_1 \mu} \frac{N_0}{\log z_1 \log y_1} \log (y_1 - y' + 1) (z' \log z' - z' - z \log z + z) . \quad . \quad (83)$$

In many cases M_0 belongs to the free end of a beam and is equal to zero.

Equation (83) might have been determined by making use of $T_2 = g\mu$, g being the angle at any point made by the trace of a vertical longitudinal plane on the neutral surface with a horizontal line. When equations (81) and (83) refer to rectangular beams, z' becomes equal to z_1. Since w and T_2 both take the value zero for $z = z'$ it follows that the depth of the beam remains the same after flexure as before for bodies of the kind of material assumed. The lateral contractions and expansions of the material at any point are just equal to the displacements due to internal tangential stresses.

There is one other source of deflection which, however, is evidently so exceedingly small in reference to the two already mentioned, that an expression for it will not be sought, though the data given are sufficient for it. This is the curved form assumed by the free-end section of the beam. If that section remains plane and normal to the axis of the beam after flexure, as has been assumed, then $\Delta_1 + w_1$ gives the total deflection. In reality, however, each point of the end section is displaced longitudinally in consequence of the distribution of the reaction in the manner already given by the general value of T_2. This third part of the deflection is due to this displacement being supposed uniformly distributed throughout the length of the beam. Such an operation would produce deflection without causing any direct stress of the kind N. Since, however, the reaction is probably never distributed in the manner indicated (the end sections therefore remaining essentially plane)

and since this part is, at best, very small compared with the others, it will be disregarded.

The total deflection of the neutral surface will then be

$$\Delta = \Delta_1 + w_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

From equation (70), combined with equation (79) or (81), according to the shape of the cross section of the beam, there result the following equations:

$$M = \frac{1}{EZ}\frac{d^2y}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

$$M = \frac{1}{EZ_1}\frac{d^2y}{dx^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

It follows from these equations that all the results of the "common theory" can be used in the application of the formulæ of this paper by simply writing $\frac{1}{Z}$ or $\frac{1}{Z_1}$ for I in those results. Consequently, the "Theorem of Three Moments" will retain precisely the same form as before, the change above indicated being only necessary.

It is much to be regretted that the form of Z is so complicated that in some cases the integration will probably be found to be impossible at present. The integration, however, will have to be performed but once for the same form of cross section. The quantity Z_1 for rectangular cross sections is of comparatively simple form, and, fortunately, by far of the greatest importance.

The form assumed by a rectangular cross section, when the beam is subjected to flexure, may easily be found in the usual manner, and requires no special attention here.

Although not strictly a part of this discussion, yet it may be interesting to notice under what conditions the assumed law for the intensity N, in the "common theory," is a true one. A beam of rectangular cross section will be considered. The notation will be the same as that used in the vicinity of equations (29–32.)

The equation of condition, which shows that the moment of resistance is constant, is

$$\int_a^{z_1} N(z - a)\, dz = \text{constant} = C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

Now, let it be required to find what law for N will make the volume of revolution $\int 2\pi N^2 dz$ a minimum, or give the following equation

$$2\pi \int_a^{z_1} N^2 dz = \text{minimum}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88)$$

Proceeding as before

$$S = \int_a^{z_1} (2\pi N^2 + a'N(z-a))\, dz\,,$$

$$\therefore V = 2\pi N^2 + a'N(z-a)\,; \quad Z = \frac{dV}{d(z-a)} = a'N\,;$$

$$Y = \frac{dV}{dN} = 4\pi N + a'(z-a)\,; \quad Y' = 0\,; \text{ \&c.}$$

Hence, from the calculus of variations,

$$4\pi N + a'(z-a) = 0 \quad \therefore N = -\frac{a'(z-a)}{4\pi}. \quad . \quad . \quad . \quad . \qquad (89)$$

The quantity a' must be such that

$$2\int_a^{z_1} -\frac{a'(z-a)^2}{4\pi}\, dz = M.$$

Equation (89) shows that *the intensity N varies directly as the distance from the neutral surface,* which is the law assumed in the "common theory" of flexure.

The law is, therefore, based on the erroneous equation (88); to be true, 2π should not appear in that equation, and N^2 should be replaced by N. $\frac{d^2V}{dN_2}$ is a positive constant, showing that equation (89) gives a value that will make V a minimum.

These last operations show that in all ordinary cases the logarithmic curve will not be a very great departure from a straight line.

It has been assumed that the coefficients of elasticity for tension and compression are equal to each other; it is easy, however, to determine the position of the neutral surface when they are not, for beams with rectangular cross sections. In Figure 6 let $ABFG$ represent the portion of a beam subjected to flexure, supposing the coefficients of elasticity to be equal to each other; the neutral surface DK will be half way between the exterior surfaces AB and GF. Now, let there be another beam $GHCF$ whose neutral and lower surfaces are coincident with those of the former, and let HC represent the upper surface of this second beam. The normal distance z_1 from DK to HC will bear such a relation to z_0, the normal distance from DK to GF, that the stress of the kind N, developed in that part of the section z_1, will be numerically equal (but of opposite sign) to that developed in the part z_0. Let E represent the smallest coefficient of elasticity and E_m the largest. From equation (71), making $M = M_1$ since any section may be taken, there results in general

$$\int_1^z N dz = \frac{N_0}{\log z_0}\left(z \log \frac{z}{e} + 1\right).$$

On account of the above assumptions, EN on one side of the neutral surface must be equal to $E_m N$ at the same distance from it on the other. The equation, therefore, which shows that the algebraic total of all the normal stresses in any section is equal to zero, is

$$E \frac{N_0}{\log z_0}\left(z_0 \log \frac{z_0}{e} + 1\right) = E_m \frac{N_0}{\log z_0}\left(z_1 \log \frac{z_1}{e} + 1\right),$$

$$\therefore E\,(z_0 \log z_0 - z_0 + 1) = E_m\,(z_1 \log z_1 - z_1 + 1). \quad . \quad . \quad . \quad . \quad (90)$$

After substituting the values of E and E_m, this transcendental equation can easily be solved by trial.

Since $E_m > E$, z_1 is of course smaller than z_0 in all cases.

This completes the strictly analytical part of the discussion, but there remains to be shown that the results are perfectly general in their character.

The general equations (2) of equilibrium were established in a manner entirely independent of the nature of the material of which the body is composed. They are three linear differential relations between six functions of the three independent variables x, y and z *only*, *i. e.*, the differentiations are in respect to those variables only. The integrations will, therefore, be made in respect to the same variables, and, in order that they may be made, there must be given certain known conditions depending on the method of application of the external forces and purely mechanical principles; these conditions are evidently entirely independent of the nature of the material. The integrations being made, the six intensities N and T will appear as functions of x, y and z only.

Again, what are known as the equations of the "tetrahedron of stress," which are simply equations (2) applied to the exterior surface of the body, are the following:

$$N_1 \cos p + T_3 \cos q + T_2 \cos r = P \cos \pi\,,$$
$$T_3 \cos p + N_2 \cos q + T_1 \cos r = P \cos \chi\,,$$
$$T_2 \cos p + T_1 \cos q + N_3 \cos r = P \cos \rho\,,$$

in which p, q and r are the angles made with the co-ordinate axes by a normal to the exterior surface at the point where the intensity P of the external force exists, and π, χ and ρ are the angles made by the direction of P with the same axes. Now, if the intensities N and T, as determined by equations (2), are functions of the nature of the material, the intensity of the externally applied force, P, is also dependent, always, on the nature of the material, which is evidently absurd. From these considerations there is deduced the

important principle, that *all problems of elastic equilibrium are completely determinate.*

It is supposed, of course, that the body has assumed its position of equilibrium; this in all ordinary cases is essentially the same as the position of no stress.

It follows immediately from the principle just enunciated that the results of this discussion are applicable to all kinds of material, whether crystalline or not, and under all degrees of stress, even up to the breaking point.

The assumption, at the beginning, of a homogeneous material with deduced results entirely independent of the nature of the material (except for deflections), emphasizes, as has been remarked, the proof of the principle first stated.

The writer regrets exceedingly being so situated that he has no apparatus at his command, otherwise the results of the preceding analysis would have been put to the test of experiment.

Data from one of the many experiments of Kirkaldy will only, therefore, be used in the moment of resistance of a rectangular beam. The bar broken was of Swedish iron two inches square, placed on supports twenty-five inches apart. The weight placed at the centre which broke the bar was 14,000 pounds. The breaking moment of the external forces at the middle section was therefore 87,500 inch pounds. The ultimate tensile resistance of the same iron was found to be about twenty-one tons (2000 pounds per ton). Consequently in equation (74) $N_0 = 21$, $b = 2$ and $z_1 = 2$. These values substituted give

$$M = 61100 \text{ inch-pounds.}$$

By the "common theory" the moment of resistance would have been only about 56000 inch-pounds. Leaving out of consideration the effects of lateral contraction and expansion, therefore, the *apparent* intensity of stress at the point of rupture would be $\frac{87500}{61100} \times 21 = 30$ tons or 60000 pounds.

It is seen from the preceding example that there is a wide discrepancy between the result of experiment and of the formulæ; of which more will be said farther on.

Figure 7 gives the results of the example graphically. Tan β is the tangent of the inclination of the curve to a vertical line at the extremity of the ordinate N. In general, as has already been shown, $\tan \beta = \frac{N_0}{\log z} \frac{1}{z}$. The depth of the beam is two inches.

$z = 1$ inch	$N = 0$	$\tan \beta = 30.3$	$\beta = 88° \ 7'$
$z = 1.25$ inches	$N = 6.76$	$\tan \beta = 24.2$	$\beta = 87° \ 38'$
$z = 1.50$ inches	$N = 12.29$	$\tan \beta = 20.2$	$\beta = 87° \ 10'$
$z = 1.75$ inches	$N = 16.96$	$\tan \beta = 17.3$	$\beta = 86° \ 41'$
$z = 2.00$ inches	$N = 21.00$	$\tan \beta = 15.15$	$\beta = 86° \ 13'$.

The scale of the figure is full size for z and for N, one twentieth of an inch for each ton, or twenty tons for each inch.

The values for β suppose one ton to the inch. They serve to show the varying inclination of the curve, but of course are not found in the figure.

The straight and dotted line shows the law of the "common theory" for the same beam, and illustrates what has been said before, that it is a moderately close approximation to the actual state of stress in a bent beam.

In regard to the discrepancy between the value of $M = 61100$ inch-pounds and the actual value determined by experiment, 87500 inch-pounds, much may be written ; but the only way by which an explanation can possibly be arrived at, is that of experiment.

In the first place equation (74) could not possibly give a result coincident with that given by Kirkaldy because in it the effect of the lateral distortion of the fibres on the value of N_0 is neglected. The support which the fibres give each other in resisting lateral contraction or expansion is believed by the writer to be the sole cause of the discrepancy between the result of the formula and that of experiment. This support could not be given were the fibres strained uniformly; in flexure, however, only those fibres equi-distant from the neutral surface are strained the same. It is known that the ultimate resistance of a bar of iron in tension is very much increased if, by any means, lateral contraction can be prevented, and the same is evidently true for compression.

The exact effect of retaining the original area of cross section can only be determined by the aid of experiments, and the writer believes that this branch of the resistance of materials offers a most fruitful and important field of experimental research, of which the limits have yet scarcely been passed.

The curve showing the intensity of stress at any point in the actual case will then probably be found to be that given in Fig. 6, but the co-ordinates representing N will have a considerable increase in length.

Plate I.

Fig. 1

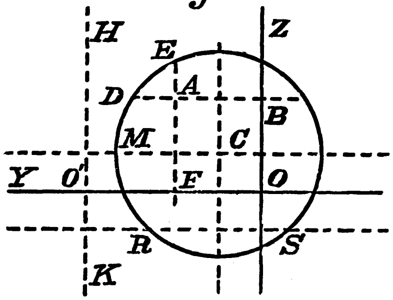


Fig. 2

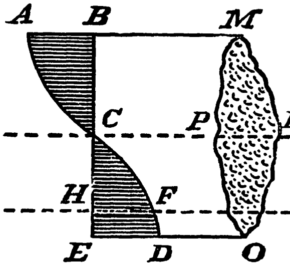


Fig. 3

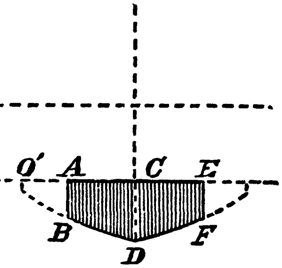


Fig. 5

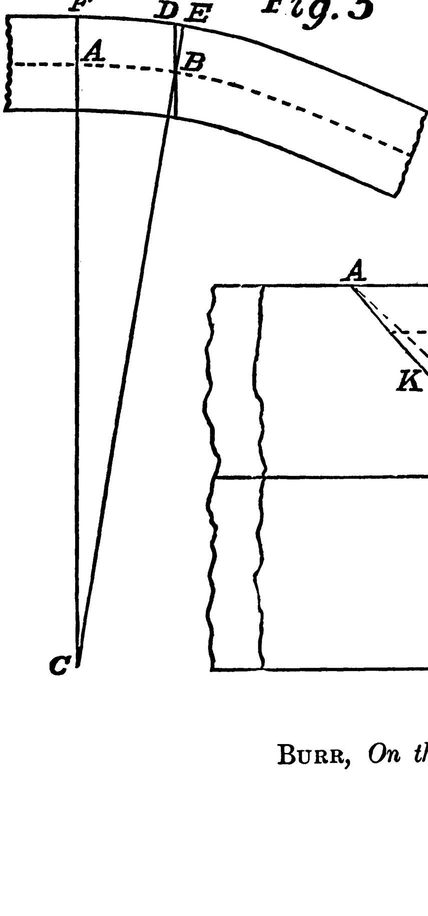


Fig. 6

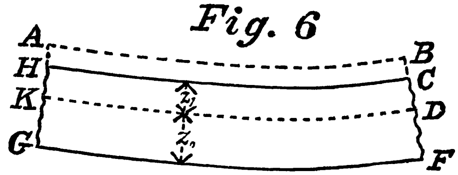
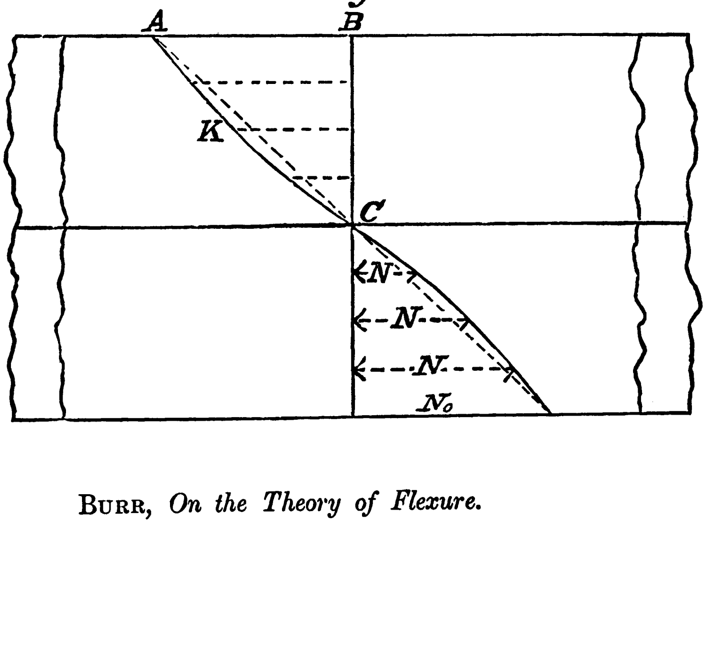


Fig. 7



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To illustrate the effect of resisting the lateral distortion of the fibres the following procedure may be employed. In equation (1), as is sometimes done, suppose $\frac{dv}{dy} = \frac{dw}{dz} = \frac{1}{4}\frac{du}{dx}$ and $\lambda = 2\mu$; then there results $N_1 = 3\mu\frac{du}{dx}$. If there is no lateral contraction then $\frac{dv}{dy} = \frac{dw}{dz} = 0$ and $N_1 = 4\mu\frac{du}{dx}$, giving an increase of $\frac{1}{3}$ over the result obtained with lateral contraction.

It is not by any means an insignificant fact that the same increase in the example taken would almost entirely make up the discrepancy observed.

Now in regard to the method by which N was established in equation (32) and those following. The principles there applied are perfectly general not being restricted to any assumptions or kind of material; they may be applied in absolutely all cases.

The restriction in the application lies in making N a function of z and y *only*, for any given section, and in the present case, as has been shown, that does not affect the generality of the results.

It is believed that the principle of least resistance has not heretofore been applied in the discussion of this problem.

It is also believed that the determinateness of the problems of elastic equilibrium has not before been so generally stated. Clebsch in his admirable work on the theory of elasticity gives a demonstration of the principle, which, however, appears to the writer to be somewhat unsatisfactory.